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ABSTRACT

This paper derives exact and asymptotic one-dimensional formulas for the probability of finding an obstacle-free interval of prescribed length. It is assumed that the obstacles correspond to randomly distributed points on the interval. The formulas relate the probability of success to size of the accessible interval and to the average density of the obstacles. The asymptotic probability is shown to be an exponential function of the accessible interval.

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SUBJECT: Probability of Finding an Obstacle - Free Interval
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DATE: August 2, 1966

FROM: P. Gunther

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TECHNICAL MEMORANDUM

INTRODUCTION

This paper derives exact and asymptotic one-dimensional formulas for the probability that the astronauts will be able to find a good touchdown interval (i.e., one free of large obstacles or craters which might damage the LM) during the terminal portion of the LM descent. The formulas are expressed as functions of the length of the accessible interval that can be searched and of the average density of obstacles. The obstacles are assumed to be randomly distributed according to a Poisson distribution. Size of the obstacles is not considered, i.e., these are assumed to be points on the search interval.

In order to properly interpret the results in terms of the probability of finding an obstacle-free touchdown area, when the accessible footprint area and average obstacle density has been specified, it is necessary to extend the solution to two dimensions. It is hoped to treat this in a subsequent paper.

The mathematical formulation of the problem is given in section I. Section II derives the basic functional equation, the exact solution to which is presented in section III and the asymptotic solution in section IV. The required size of the accessible interval vs. obstacle density, for various specified levels of confidence, is plotted in Figure 3. Appendix B presents a mathematical proof that the required probability can be expressed as an infinite sum of the residues at the poles of the Laplace transform. The poles are discussed in Appendix A.

I. FORMULATION OF PROBLEM

Let H be the length of the (maximum) search interval, and ℓ the subinterval length required to be obstacle-free. The obstacles are assumed to be Poisson distributed with density parameter λ, i.e., λ is the expected number of obstacles per unit length. As is well known, the Poisson distribution implies that the probability that a single obstacle appears in the infinitesimal length ds is λds, and the probability of multiple obstacles appearing in ds is negligible (of higher order). Within any finite interval of length s, the probability of finding r obstacles is

$$\text{(1)} \qquad \text{Prob}(r) = e^{-\lambda s}(\lambda s)^r/r! \quad .$$

It is required to determine the probability Q(H) that every interval of length ℓ within the search interval (0,H) contains at least one obstacle. 1-Q(H) is then the probability that at least one obstacle-free interval of length ℓ exists.

Instead of treating H as fixed, it will be convenient to view the obstacles as arising sequentially as the search interval is traversed. This is similar to the approach adopted in study of waiting time processes. From this point of view, the Poisson distribution of events goes over into the exponential "waiting distance" distribution with probability density

$$\text{(2)} \qquad f(s) = \lambda e^{-\lambda s} \quad , \qquad s > 0$$

The corresponding cumulative probability that the waiting distance is less than s is $1-e^{-\lambda s}$. Hence, when $H=\ell$ the probability q(say) that the interval $(0,\ell)$ is not obstacle-free is

$$\text{(3)} \qquad q = 1 - e^{-\lambda \ell}$$

The boundary conditions for the problem which, loosely speaking, serve to define the edge effects will be given by

$$(4) \qquad Q(H) = q_0 \quad , \qquad 0 \leq H < \ell$$

Ordinarily, one would take $q_0=1$, since for $\ell>H$ one cannot <u>know</u> that the interval $(0,\ell)$ will be free of obstacles. However, other interpretations are possible, e.g., if $Q(H)$ is considered to include the subsequent action of aborting or of attempting a landing anyway.

One can obtain a useful simplification through the transformation:

$$(5) \qquad \begin{aligned} H &\rightarrow H/\ell \\ \lambda\ell &\rightarrow \lambda \end{aligned}$$

This is equivalent to assuming $\ell=1$, so that H is measured in units of the subinterval length ℓ, and λ becomes expected number of obstacles in ℓ.

The problem formulated above can be considered as the continuous analog of the discrete problem (Feller, Ref. 1, pp. 260-261) of finding the probability of no success run of length r in n independent trials, where p is the probability of success on any single trial. The correspondence is given by

$$\begin{aligned} p &= 1-\lambda ds \\ n &= H/ds \\ r &= \ell/ds \end{aligned}$$

In Section IV, Feller's asymptotic solution to the discrete problem is extended to the continuous case.

II. DERIVATION OF FUNCTIONAL EQUATION

Consider the event that there is no obstacle-free subinterval of length 1 in (0,H). If in the first unit interval an obstacle occurs at $s(0 \le s < 1)$, then there must be no obstacle-free unit length interval in the remaining search interval of length H-s. This observation leads to the following integral equation which describes the recursive nature of Q(H).

$$(6) \qquad Q(H) = \int_0^1 Q(H-s)\lambda e^{-\lambda s} ds$$

$$= e^{-\lambda H} \int_{H-1}^{H} Q(t)\lambda e^{\lambda t} dt \quad , \qquad H \ge 1$$

The boundary condition for (6) is (cf. (4))

$$(7) \qquad Q(H) = q_0 \; , \qquad 0 \le H < 1$$

It is easy to see that Q(H) is continuous for H>1. For H=1, we define

$$(8) \qquad Q(1) \equiv Q(1+0) \equiv q_1$$

It then follows from (6) that

$$(9) \qquad q_1 = q_0(1 - e^{-\lambda})$$

It will be shown below that Q(H) is differentiable for $H \neq 1,2$. (In these exceptional cases we define $Q'(1) = Q'(1+0)$, and $Q'(2) = Q'(2+0)$.) Equation (6) then implies that

$$(10) \qquad Q'(H) = \int_0^1 Q'(H-s)\lambda e^{-\lambda s} ds \ , \qquad H \geq 1$$

Integrating by parts gives

$$Q'(H) = -\, Q(H-1)\lambda e^{-\lambda} + Q(H)\lambda - \lambda \int_0^1 Q(H-s)\lambda e^{-\lambda s} ds$$

By (6), the integral on the right is simply Q(H), so that (10) leads to the following difference-differential equation

$$(11) \qquad Q'(H) = b\, Q(H-1) \ , \qquad H > 1, \quad H \neq 2$$

where

$$(12) \qquad b = -\, \lambda e^{-\lambda}$$

Equation (11) is solved in the next section, treated as a formal mathematical problem devoid of its probability interpretation. The solution is expressed as a function of arbitrary boundary parameters q_0 and q_1 in (7) and (8), and of the parameter b.

III. EXACT SOLUTION

Integrating (11) between the limits 1 and H gives

$$(13) \qquad Q(H) = Q(1) + b \int_1^H Q(s-1)ds$$

$$= q_1 + b \int_0^{H-1} Q(s)ds \ , \qquad H \geq 1$$

Thus Q(H) can be determined recursively for successive unit intervals. For example,

$$
(14)\qquad
\begin{aligned}
0 \le H < 1 :&\quad Q(H) = q_0 \\
1 \le H < 2 :&\quad Q(H) = q_1 + bq_0(H-1) \\
2 \le H < 3 :&\quad Q(H) = q_1 + b\int_0^1 q_0 ds + b\int_1^{H-1} [q_1+bq_0(s-1)]ds \\
&\qquad\quad = q_1 + bq_0 + bq_1(H-2) + \frac{b^2}{2} q_0(H-2)^2
\end{aligned}
$$

In general, for $m \le H < m + 1$, Q(H) is easily seen, by induction, to be a polynomial of degree m in H-m, i.e.,

$$
(15)\qquad Q(H) = \sum_{n=0}^{m} a_{mn}(H-m)^n \qquad m = 0,1,\cdots
$$

The following notation will simplify the derivation of the coefficients a_{mn}. Let

$$
H \equiv m + h \quad , \qquad 0 \le h < 1
$$

where m=[H], the largest integer $\le$ H. We define also

$$
(16)\qquad Q_m(h) \equiv Q(H) = \sum_{n=0}^{m} a_{mn}h^n \ , \qquad m=0,1,\cdots \ , \quad 0 \le h < 1
$$

In particular,

(17) $$Q_m(0) \equiv Q(m) \equiv q_m = a_{m0} \quad , \quad m=0,1,\cdots$$

Since Q(H) is continuous for H > 1,

(18) $$Q_{m-1}(1) = Q_m(0) = q_m = a_{m0} \; , \quad m=2,3,\cdots$$

Setting h=1 in (16) with m = m-1 gives

(19) $$q_m = a_{m0} = \sum_{n=0}^{m-1} a_{m-1,n}$$

The functional equation (11) can now be written

(20) $$Q_m'(h) = bQ_{m-1}(h)$$

the boundary conditions being

$$Q_0(h) = q_0$$

$$Q_1(0) = q_1$$

Integrating (20) between 0 and h gives

$$(21)\quad Q_m(h) = Q_m(0) + b\int_0^h Q_{m-1}(h)dh$$

$$= Q_m(0) + b\int_0^h dh_1\left\{Q_{m-1}(0)+b\int_0^{h_1} dh_2\left[Q_{m-2}(0)+\cdots+b\int_0^{h_{m-1}} dh_m Q_0(0)\right]\right\}$$

$$= \sum_{n=0}^{m} Q_{m-n}(0)\ \frac{(bh)^n}{n!}$$

or

$$(22)\quad Q_m(h) = \sum_{n=0}^{m} q_{m-n}\ \frac{(bh)^n}{n!}$$

Setting h=1 gives, from (18)

$$(23)\quad q_{m+1} = \sum_{n=0}^{m} q_{m-n}\ \frac{b^n}{n!} \quad , \qquad m=1,2,\cdots$$

Also, substituting (16) into (21) and equating powers of h gives

$$(24)\quad a_{mn} = a_{m-n,n-1}\cdot b/n$$

$$= a_{m-n,0}\ b^n/n! = q_{m-n}\ b^n/n!$$

The recursive relations, (19) and (24), imply that a_{mn} can be computed using the scheme illustrated by the matrix (25) on the following page. The $0^{\underline{th}}$ entry in the $m^{\underline{th}}$ row, namely

(25)

	0	1	2	3	4
q_0:	q_0				
q_1:	q_1	$q_0 b$			
q_2:	$q_1 + q_0 b$	$q_1 b$	$q_0 \frac{b^2}{2!}$		
q_3:	$q_1 + (q_0+q_1)b + q_0 \frac{b^2}{2!}$	$q_1 b + q_0 b^2$	$q_1 \frac{b^2}{2!}$	$q_0 \frac{b^3}{3!}$	
q_4:	$q_1+(q_0+2q_1)b+(3q_0+q_1)\frac{b^2}{2!}+q_0\frac{b^3}{3!}$	$q_1 b+(q_0+q_1)b^2+q_0\frac{b^3}{2!}$	$q_1\frac{b^2}{2!}+q_0\frac{b^3}{2!}$	$q_1\frac{b^3}{3!}$	$q_0\frac{b^4}{4!}$

$q_m = a_{m0}$, is obtained by summing all elements in the preceding row (equation (19)). The $n^{\underline{th}}$ entry in the $m^{\underline{th}}$ row is obtained by multiplying the preceding adjacent (left upper) diagonal element by b/n (equation (24)), or by multiplying the $0^{\underline{th}}$ element belonging to the diagonal by $b^n/n!$ Note that the terms in the main and in the first off-diagonal are identical except for the multiplier coefficient, q_0 or q_1, respectively.

$Q_m(h)$ can then be determined either from (16) by multiplying successive elements in the $m^{\underline{th}}$ row by h^n, or from (22) by multiplying successive elements in the $0^{\underline{th}}$ column by $(bh)^n/n!$

An explicit formula can be derived for the q_m. For this purpose, in order to emphasize the dependence of q_m on the parameters q_0, q_1, b we write $q_m \equiv q_m(b;q_0,q_1)$.

Equation (23) is then written

$$q_{m+1}(b;q_0,q_1) = \sum_{n=0}^{m} q_{m-n}(b;q_0,q_1)\, \frac{b^n}{n!} \tag{26}$$

We observe from the matrix (25) that the polynomials q_m are linear in q_0 and q_1. In fact,

$$q_m(b;1,1) = q_{m+1}(b;0,1) \tag{27}$$

$$q_m(b;q_0,q_1) = q_m(b;q_0,q_0) + q_m(b;0,q_1-q_0) =$$

$$q_0 \cdot q_m(b;1,1) + (q_1-q_0)\, q_m(b;0,1) = \tag{28}$$

$$q_0 \cdot q_{m+1}(b;0,1) + (q_1-q_0)\, q_m(b;0,1)$$

For simplicity we write

$$q_m(b;0,1) = q_m(b)$$

We have then the following lemma.

Lemma: Let

$$\text{(29a)} \qquad q_{m+1}(b) = \sum_{n=0}^{m} q_{m-n}(b)\,\frac{b^n}{n!} \quad , \qquad m = 1,2,\cdots$$

$$q_0(b) = 0$$

$$q_1(b) = 1$$

Then

$$\text{(29)} \qquad q_{m+1}(b) = \sum_{n=0}^{m} \frac{(m-n)^n}{n!}\,b^n \quad , \qquad m = 0,1,\cdots$$

Proof: The proof is by induction. For m=1, the right hand sides of (29) and (29a) are 1. (Equation (29) holds also for m=0 if the convention $0^0=1$ is adopted.) Suppose (29) holds for $m \leq k$. Then

$$q_{k+2}(b) = \sum_{n=0}^{k} q_{k-n+1}(b) \frac{b^n}{n!} = \sum_{n=0}^{k} \left(\sum_{j=0}^{k-n} \frac{(k-n-j)^j}{j!} b^j \right) \frac{b^n}{n!} =$$

$$= \sum_{n=0}^{k} \sum_{j=0}^{k-n} \frac{(k-n-j)^j}{j!n!} b^{n+j} = \sum_{n=0}^{k} \sum_{i=n}^{k} \frac{(k-i)^{i-n}}{n!(i-n)!} b^i =$$

$$= \sum_{i=0}^{k} \sum_{n=0}^{i} \frac{(k-i)^{i-n}}{n!(i-n!)} b^i = \sum_{i=0}^{k} \frac{(1+k-i)^i}{i!} b^i$$

This proves the lemma.

Substituting (29) into (28) gives the final solution

$$(30) \qquad q_{m+1} = \sum_{n=0}^{m} \left[q_0(m-n+1)^n + (q_1-q_0)(m-n)^n \right] \frac{b^n}{n!} \qquad m = 1,2,\cdots$$

An alternative derivation of (30) is through the use of generating functions. Let

$$(31) \qquad F(z) \equiv \sum_{m=0}^{\infty} q_m z^m$$

Multiplying (23) by z^m and summing from 1 to ∞ gives

$$(32) \qquad \sum_{m=1}^{\infty} q_{m+1} z^m = \sum_{m=1}^{\infty} \sum_{n=0}^{m} q_{m-n} \frac{b^n}{n!} z^m$$

The left hand side of (32) is equal to

$$\frac{1}{z} (F(z) - q_0 - q_1 z)$$

while the right hand side, after interchanging the order of summation, is

$$- q_0 + \sum_{n=0}^{\infty} \left[\sum_{m=n}^{\infty} q_{m-n} z^{m-n} \right] \frac{(bz)^n}{n!} = - q_0 + F(z) \cdot e^{bz}$$

Equating these gives

$$F(z) = \frac{q_0 + (q_1 - q_0)z}{1 - ze^{bz}} \tag{33}$$

Expanding (33) into a power series in z gives for q_m, the coefficient of z^m, precisely the formula (30).

IV. ASYMPTOTIC SOLUTION FOR Q(H)

In this final section, we treat only the special case (cf. sections I and II) of $q_0=1$, $q_1 = 1 - e^{-\lambda}$, $b = - \lambda e^{-\lambda}$. Equation (30) then specializes to

$$q_{m+1} = \sum_{n=0}^{m} \left[(m-n+1)^n - e^{-\lambda}(m-n)^n \right] \frac{(-\lambda e^{-\lambda})^n}{n!} \tag{34}$$

and equation (33) becomes

$$F(z) = \frac{1 - ze^{-\lambda}}{1 - ze^{-z\lambda e^{-\lambda}}} \tag{35}$$

$$\equiv \frac{N(z)}{D(z)}$$

where N(·) represents the numerator of F(z) and D(·) the demoninator.

A formula for q_m can be derived in terms of the residues of the poles of F(z), where z is now considered to lie in the complex plane. It is shown in Appendix B that Feller's formula (reference 1, p. 258, equation (4.8)) can be extended to the continuous case. Using only the smallest pole (in absolute value) then provides an asymptotic approximation for large H. More specifically, let z_0 be the smallest root (in absolute value) of the denominator of F(z) in (35). Then, asymptotically,

$$q_m \sim -\frac{N(z_0)}{D'(z_0)} z_0^{-(m+1)} \tag{36}$$

where the sign $\sim$ indicates that the ratio of the two sides tends to unity. z_o is in fact the unique real positive solution of the equation D(z) = 0, i.e., of

$$z_0 = e^{z_0 \lambda e^{-\lambda}} = e^{-bz_0} \tag{37}$$

(The root $z_0=e^{\lambda}$ is excluded, since it is not a pole of F(z).) For $\lambda \neq 1$, z_0 can be seen to be a simple root and is also the smallest in absolute value among all complex roots (see Appendix A.) Now

$$D'(z_0) = -e^{bz_0}(1+bz_0) = -\frac{1+bz_0}{z_0} \tag{38}$$

Substituting (38) into (36) gives then

$$q_m \sim z_0^{-m} \cdot \frac{1-z_0 e^{-\lambda}}{1-z_0 \lambda e^{-\lambda}} \tag{39}$$

The asymptotic expression (39) actually holds for non-integral values of H as well. This can be seen by considering the Laplace transform for Q(H). The analysis is similar to that for the generating function F(z). Let

$$(40) \qquad \phi(u) \equiv \int_0^\infty Q(H)e^{-uH}dH$$

Multiplying equation (11) by e^{-uH}, integrating between 1 and ∞, and then using integration by parts, gives

$$-q_1e^{-u} + u\phi(u) - q_0(1-e^{-u}) = be^{-u}\phi(u)$$

Substituting for the parameters in terms of λ, and defining for simplicity $v \equiv -u$ and $\phi(-v) \equiv \psi(v)$, gives

$$(41) \qquad \psi(v) = \frac{e^{-v} - e^{-\lambda}}{\lambda e^{-\lambda} - ve^{-v}}$$

Let v_0 be the unique real root ($\neq\lambda$) of the equation

$$(42) \qquad v_0e^{-v_o} = \lambda e^{-\lambda}$$

If $\lambda>1$, then $v_0<1$; while if $\lambda<1$, then $v_0>1$. The asymptotic formula analogous to (36) then becomes (where $N(\cdot)$ and $D(\cdot)$ now refer to (41)):

$$Q(H) \sim - e^{-v_0 H} \cdot \frac{N(v_0)}{D'(v_0)} \tag{43}$$

or

$$Q(H) \sim e^{-v_0 H} \cdot \frac{1 - v_0/\lambda}{1 - v_0} \quad , \qquad \text{for } \lambda \neq 1 \tag{44}$$

Note that $z_0 = e^{v_o}$ so that (44) is identical with (39) when H=m.

When λ=1, (41) can be written as

$$\psi(v) = \frac{1 - e^{v-1}}{e^{v-1} - v} = - \frac{1 + \frac{v-1}{2!} + \frac{(v-1)^2}{3!}}{\frac{v-1}{2!} + \frac{(v-1)^2}{3!} + \cdots} \tag{45}$$

Hence v_0=1 is seen to be a simple pole, and is in addition the smallest root in absolute value of the denominator (see Appendix A). Since from (45) D'(1) = 1/2 and N(1) = 1, equation (43) becomes

$$Q(H) \sim 2e^{-H} \quad , \qquad \text{for } \lambda = 1 \tag{46}$$

The solution of (42) can be obtained using the following iteration:

$$v_{n+1} = \lambda e^{-\lambda} e^{v_n} \tag{47}$$

This converges only for $\lambda>1$ and initial value $v_1<\lambda$. When λ is large, two iterations starting with $v_1=0$ results in a fairly good approximation for v_0, namely

$$(48) \qquad v_0 \simeq \lambda e^{-\lambda} e^{\lambda e^{-\lambda}}$$

Newton's method is somewhat more efficient and leads to the following iteration

$$(49) \qquad v_{n+1} = \frac{1 - v_n}{1 - \lambda e^{-\lambda} e^{v_n}} \cdot \lambda e^{-\lambda} e^{v_n}$$

This converges for $\lambda \gtrless 1$, but the convergence is more rapid for $\lambda>1$. In this case the starting value should satisfy $v_1 < \lambda - \ln \lambda$.

For $\lambda<1$ a simpler and more rapid iteration than (49) is

$$(50) \qquad v_{n+1} = \lambda - \ln \lambda + \ln v_n$$

with $v_1>\lambda$. The corresponding Newton iteration is

$$(51) \qquad v_{n+1} = \frac{v_n}{v_n-1} [\lambda - \ln \lambda - 1 + \ln v_n]$$

This also converges for $\lambda \gtrless 1$, but is more efficient for $\lambda<1$ with $v_1 > 1$.

It is convenient to re-write equations (44) and (46) in the following form, where the solution v_0 of (42) is now written as $v(\lambda)$:

(52) $$Q(H) \sim [1 + K(\lambda)]e^{-v(\lambda)H}$$

where

(53) $$K(\lambda) = \begin{cases} \frac{1-\lambda}{\lambda} \cdot \frac{v(\lambda)}{1-v(\lambda)} & \text{for } \lambda \neq 1 \\ 1 & \text{for } \lambda = 1 \end{cases}$$

Figure 1 plots $v(\lambda)$ and $K(\lambda)$ on semilog paper. The curves are actually slightly S-shaped with asymptotic slope of -1 (on a natural-log ordinate scale).

Figure 2 plots Q(H) vs. H for various λ and compares the exact solution given by (22) and (34) with the asymptotic solution given by (52). It can be seen that the approximation is excellent for $H > 1.5$, even for small λ. Although we have not proved this property, it appears that for integral H, the asymptotic formula for q_m constitutes an upper bound of all m.

In order to obtain equiprobability (or confidence) contours of (required) H vs. λ, we set $P=1-Q$ and $H=H_P$. Solving equation (52) for H_P gives approximately

(54) $$H_P \sim \frac{1}{v(\lambda)} \ln \frac{1+K(\lambda)}{1-P}$$

Figure 3 plots (54) for P = .50, .90, .95, .99 and .999. The portion of the curves for small λ have been determined from the exact solutions plotted in Figure 2.

As was noted in section I, H can be interpreted as the random waiting interval until an obstacle-free unit interval arises. 1-Q(H) is then the cumulative probability distribution, the derivative of which yields the probability density function for waiting intervals.* The mean waiting time and variance is easily determined from the Laplace transform (41). In fact, using well known formulas,

(55) $$E(H) \equiv \psi(0) = \frac{1 - e^{-\lambda}}{\lambda e^{-\lambda}}$$

(56) $$V(H) \equiv 2\psi'(0) - \psi^2(0) = \frac{1}{(\lambda e^{-\lambda})^2} - \frac{2}{\lambda e^{-\lambda}} - \frac{1}{\lambda^2}$$

It is of interest to compare these results with the approximate distribution (represented hereafter by the symbol $\sim$). For illustrative purposes, we restrict attention to the case of $\lambda=1$. In order that $Q(H)$ represent a cumulative distribution, it is necessary that $Q(0)=1$. More specifically, equation (46) is modified for $H<1$ to give

$$\tilde{Q}(H) = \begin{cases} 1 & \text{for } 0 \leq H < 1 \\ 2e^{-H} & \text{for } H \geq 1 \end{cases}$$

Then

$$\tilde{E}(H) = \int_0^\infty \tilde{Q}(H)dH = 1+2/e = 1.736$$

$$\tilde{V}(H) = 2\int_0^\infty H\tilde{Q}(H)dH - \tilde{E}^2(H) = 4(e-1)/e^2 = .931$$

---

*To obtain the exact probability density function in the manner of the recursive scheme (25), the 0th column should be eliminated and the nth column multiplied by n.

The exact values, obtained by substituting $\lambda=1$ into (55) and (56), are

$$E(H) = e - 1 = 1.718$$

$$V(H) = e^2 - 2e - 1 = .9525$$

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M. J. Norris, formerly of Bellcomm, made many significant contributions to this paper, particularly in the derivation of the exact solution presented in section II.

P Gunther

2023-PG-gdn P. Gunther

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APPENDIX A

POLES OF THE LAPLACE TRANSFORM OF Q(H)

From equation (41) for $\psi(v)$ (which is equivalent to the usual Laplace transform $\phi(u)$ defined by (40)) the poles consist of the solutions of equation (42):

$$\lambda e^{-\lambda} - ve^{-v} = 0 \tag{A1}$$

The single real solution ($\neq\lambda$) is v_0 and the complex solutions can be written as

$$v = x + iy \tag{A2}$$

where x and y are real and $y\neq0$. Substituting (A2) into (A1) gives

$$\lambda e^{-\lambda} - (x+iy)e^{-x}\, e^{-iy} = 0 \tag{A3}$$

Separating into real and imaginary parts yields the two equations

$$R \equiv \lambda e^{-\lambda} - e^{-x}(x \cos y + y \sin y) = 0 \tag{A4}$$

$$I \equiv e^{-x}(x \sin y - y \cos y) = 0 \tag{A5}$$

These equations imply

$$x = y \cot y \tag{A6}$$

$$\lambda e^{-\lambda} = e^{-x}\, x \sec y \tag{A7}$$

$$= e^{-x}\, x \csc y \,. \tag{A8}$$

(A1) and (A7) imply that the complex poles are unchanged if v_0 and λ are interchanged. Moreover, (A6) and (A8) show that if

$x+iy$ is a pole then so also is $x-iy$. Hence, we can assume hereafter that $y>0$. (A8) then implies that $\csc y>0$, so that y is either in the first or second quadrants. Actually, only the first quadrant is allowable. First note that x is positive since

$$\text{(A9)} \qquad x = \lambda - \ln \lambda + \ln(y \csc y) > \lambda - \ln \lambda \geq 1$$

Equation (A7) now implies that $\sec y>0$ so that y cannot be in the second quadrant. Moreover, (A9) can be sharpened to

$$\text{(A10)} \qquad x > \max\,(\lambda, v_0)$$

since, by (A6), (A8), and (A1),

$$x - \ln x = y \cot y - \ln(y \cot y) > y \cot y - \ln(y \csc y) =$$

$$= \lambda - \ln \lambda = v_0 - \ln v_0$$

(A10) then follows from (A9). Hence, v_0 is the smallest pole in absolute value.

The complex poles can be represented as $v_n=x_n+iy_n$, where

$$\text{(A11)} \qquad y_n = 2\pi n + y_n' \quad , \quad n = 1, 2, \cdots$$

$$\text{(A12)} \qquad 0 < y_n' < \pi/2$$

There is no solution for $n=0$, since (A6) would then imply that $x_0=y_0' \cot y_0'<1$ which contradicts (A9). x_n and y_n' are obtained from (A8) and (A6) as the solution of the equations

$$\text{(A13)} \qquad (2\pi n+y_n')\cot y_n' - \ln[(2\pi n+y_n')\csc y_n'] = \lambda - \ln \lambda$$

$$\text{(A14)} \qquad x_n = (2\pi n+y_n')\cot y_n'$$

Equation (A13) implies that as $n \to \infty$

(A15) $$\lim y_n' = \frac{\pi}{2}$$

For large n, equations (A13)-(A15) then imply that

(A16) $$x_n \sim \lambda - \ln \lambda + \ln(2n+\tfrac{1}{2})\pi$$

Figure B1 shows the asymptotic location of the poles, corresponding to (A15) and (A16), for $\lambda = 1.5$.

A more accurate estimate of the poles can be obtained using an approximation derived from a first order expansion of (A13) about $\pi/2$. Letting

(A17) $$z_n' \equiv \pi/2 - y_n'$$

then

(A18) $$z_n' \simeq \frac{\lambda-\ln \lambda+\ln(2n+\frac{1}{2})\pi}{(2n+\frac{1}{2})\pi}$$

Also,

(A19) $$x_n = \lambda - \ln \lambda + \ln(2n+\tfrac{1}{2})\pi + \ln\left(1 - \frac{z_n'}{(2n+\frac{1}{2})\pi}\right) - \ln \cos z_n'$$

(20) $$\simeq \lambda - \ln \lambda + \ln(2n+\tfrac{1}{2})\pi - \frac{z_n'}{(2n+\frac{1}{2})\pi} + \frac{z_n'^2}{2}$$

For $\lambda=1.5$ and $n=1$, the exact solution from (A13) and (A14) is $z_1'=.404$, $x_1=3.187$. The approximate solution from (A18) and (A20) is $z_1'\simeq.402$, $x_1\simeq3.185$. The asymptotic solution (A16) gives $x_1=3.156$.

APPENDIX B

PROOF OF ASYMPTOTIC FORMULA FOR Q(H)

Since Q(H) is integrable, i.e., EH exists, and has everywhere right and left sided derivatives, the Laplace transform can be inverted to give

$$Q(H) = \lim_{R_y \to \infty} \frac{1}{2\Pi} \int_{-iR_y}^{+iR_y} e^{-vH} \psi(v) dv \tag{B1}$$

At points of discontinuity (H=0,1), $Q(H) = \frac{1}{2}[Q(H+0) + Q(H-0)]$.

The integral in (B1) can be evaluated by contour integration. Using the rectangular contour shown in Figure B1, with boundary C_1, C_2, C_3, C_4, it is well known that, for the 2n+1 poles enclosed within the boundary

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 2\Pi \sum_{-n}^{n} \rho_i \tag{B2}$$

Each integral on the left is the same as in (B1). ρ_i, the residue at the pole v_i is given by (cf. equation (44))

$$\rho_i = e^{-v_i H} \cdot \frac{1-v_i/\lambda}{1-v_i} \tag{B3}$$

We will show that as $n \to \infty$

$$\lim \int_{C_2} = \lim \int_{C_3} = \lim \int_{C_4} = 0 \tag{B4}$$

Then, since $\ell im \int_{C_1}$ corresponds to the integral in (B1), it will follow that

(B5) $$Q(H) = \sum_{-\infty}^{\infty} \rho_i$$

$$= \rho_0 + \frac{2}{\lambda} \sum_{1}^{\infty} e^{-x_i H} \left[\frac{(x_i-\lambda)(x_i-1)\cos y_i H + (\lambda-1) y_i \sin y_i H}{(x_i-1)^2 + y_i^{\,2}} \right]$$

Moreover, since v_0 is the pole with smallest absolute value, then asymptotically for large H, it is easy to see that

(B6) $$Q(H) \sim \rho_0$$

The sides of the rectangle in Figure B1 are taken such that

$$R_{x_n} = R_{y_n} = (2n+1)\Pi$$

In addition C_2 is subdivided into $C_2^{'}$ and $C_2^{''}$, with the length of $C_2^{'}$ given by

$$R_{x_n}^{'} = \ell n\ \ell n(2n+1)\Pi$$

To show that $\ell im \int_{C_2^{'}} = 0$ it suffices to show that $\psi(v) \to 0$ on $C_2^{'}$. For sufficiently large n, since $0 \le x \le R_{x_n}^{'}$

$$\left|ve^{-v} - \lambda e^{-\lambda}\right| \geq \left|(x+iR_{y_n})e^{-x-iR_{y_n}}\right| - \lambda e^{-\lambda} = \sqrt{x^2+R_{y_n}^2}\, e^{-x} - \lambda e^{-\lambda} \geq$$

$$\geq R_{y_n} e^{-R'_{y_n}} - \lambda e^{-\lambda} = \frac{(2n+1)\Pi}{\ell n(2n+1)\Pi} - \lambda e^{-\lambda}$$

Since $\left|e^{-\lambda} - e^{-v}\right| \leq 2$, it follows that

$$|\psi(v)| = \frac{\left|e^{-\lambda} - e^{-v}\right|}{\left|ve^{-v} - \lambda e^{-\lambda}\right|} \to 0$$

For C_2'', equations (A4) and (A5) with $y=(2n+1)\Pi$ give

$$\left|\lambda e^{-\lambda} - ve^{-v}\right| = \left|\lambda e^{-\lambda} + xe^{-x} + i(2n+1)\Pi e^{-x}\right| \geq \lambda e^{-\lambda}$$

uniformly in n. Then

$$\left|\int_{C_2''}\right| \leq \frac{2}{\lambda e^{-\lambda}} \int_{\ell n\, \ell n(2n+1)\Pi}^{(2n+1)\Pi} e^{-xH} dx =$$

$$= \frac{2}{H\lambda e^{-\lambda}} \left[(\ell n(2n+1)\Pi)^{-H} - e^{-(2n+1)\Pi H}\right]$$

so that $\ell im \int_{C_2''} = 0$. Hence, $\ell im \int_{C_2} = 0$. In similar fashion $\ell im \int_{C_4} = 0$.

For C_3, with $-(2n+1)\pi < y < (2n+1)\pi$ and $R_{x_n} = (2n+1)\pi$, then for sufficiently large n

$$\left|\lambda e^{-\lambda} - ve^{-v}\right| \geq \lambda e^{-\lambda} - \left|R_{x_n} + iy\right| e^{-R_{x_n}} = \lambda e^{-\lambda} -$$

$$- \sqrt{R_{x_n}^2 + y^2}\; e^{-R_{x_n}} \geq \lambda e^{-\lambda} - \sqrt{2}\, R_{x_n} e^{-R_{x_n}} \geq \frac{1}{2} \lambda e^{-\lambda}$$

uniformly in n. Hence,

$$\left| \int_{C_3} \right| \leq 2(2n+1)\pi \cdot \frac{2}{\frac{1}{2}\lambda e^{-\lambda}} \; e^{-(2n+1)\pi}$$

so that $\lim_n \int_{C_3} = 0$. This completes the proof of equations (B4) and hence of (B5).

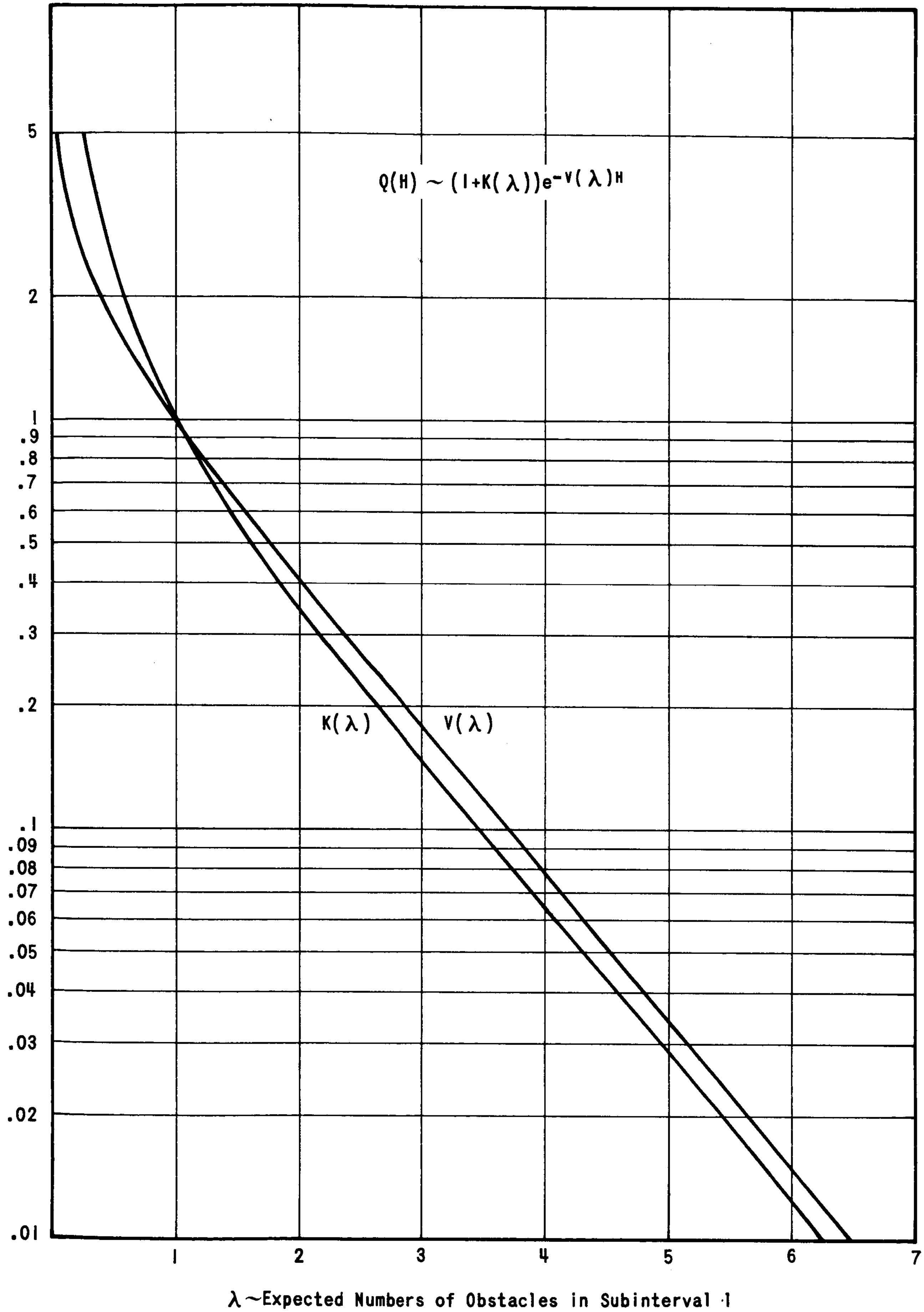

FIGURE 1 - CONSTANTS FOR ASYMPTOTIC APPROXIMATION TO Q(H)

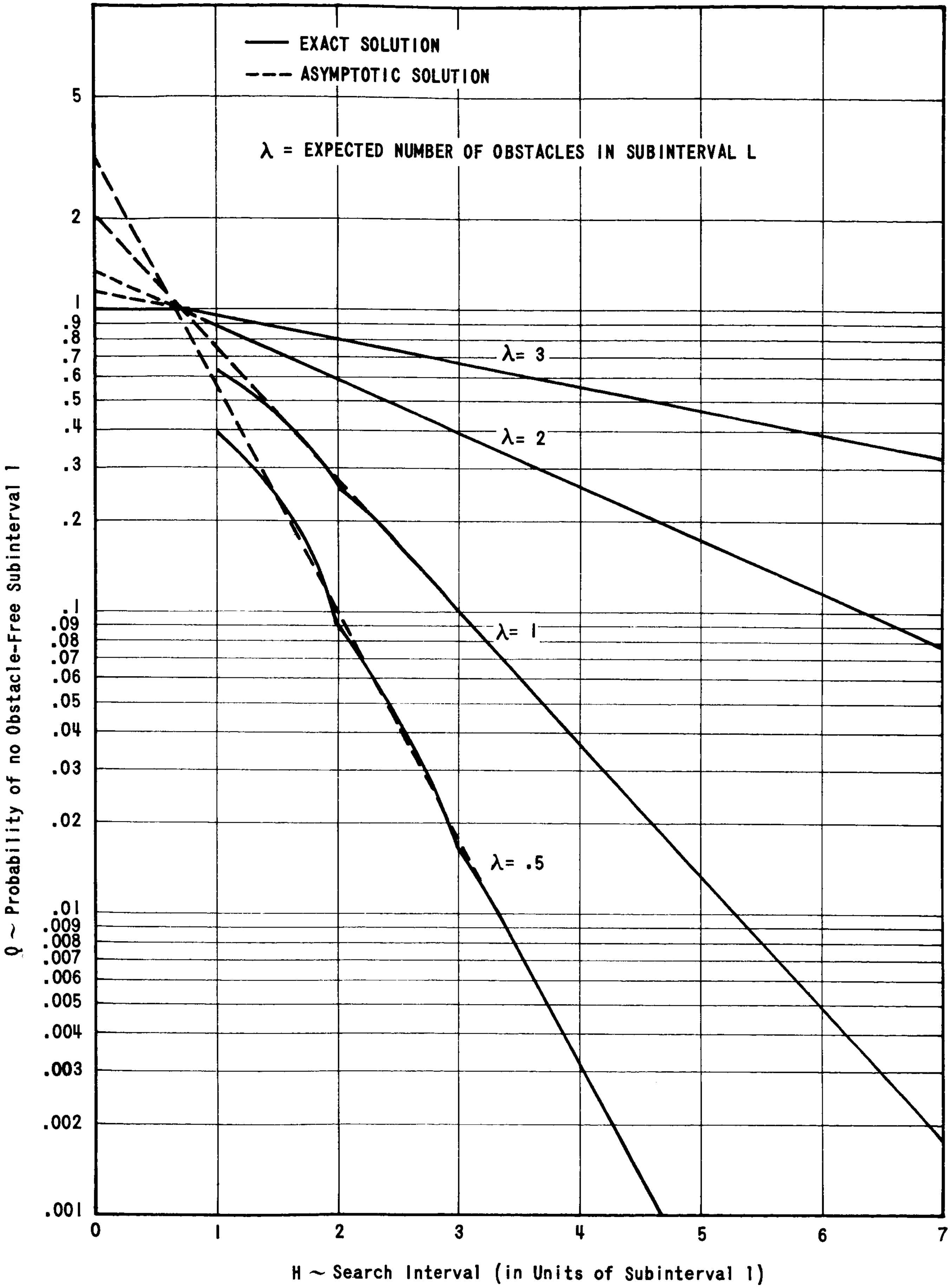

FIGURE 2 - COMPARISON OF EXACT AND APPROXIMATE PROBABILITY FORMULAE

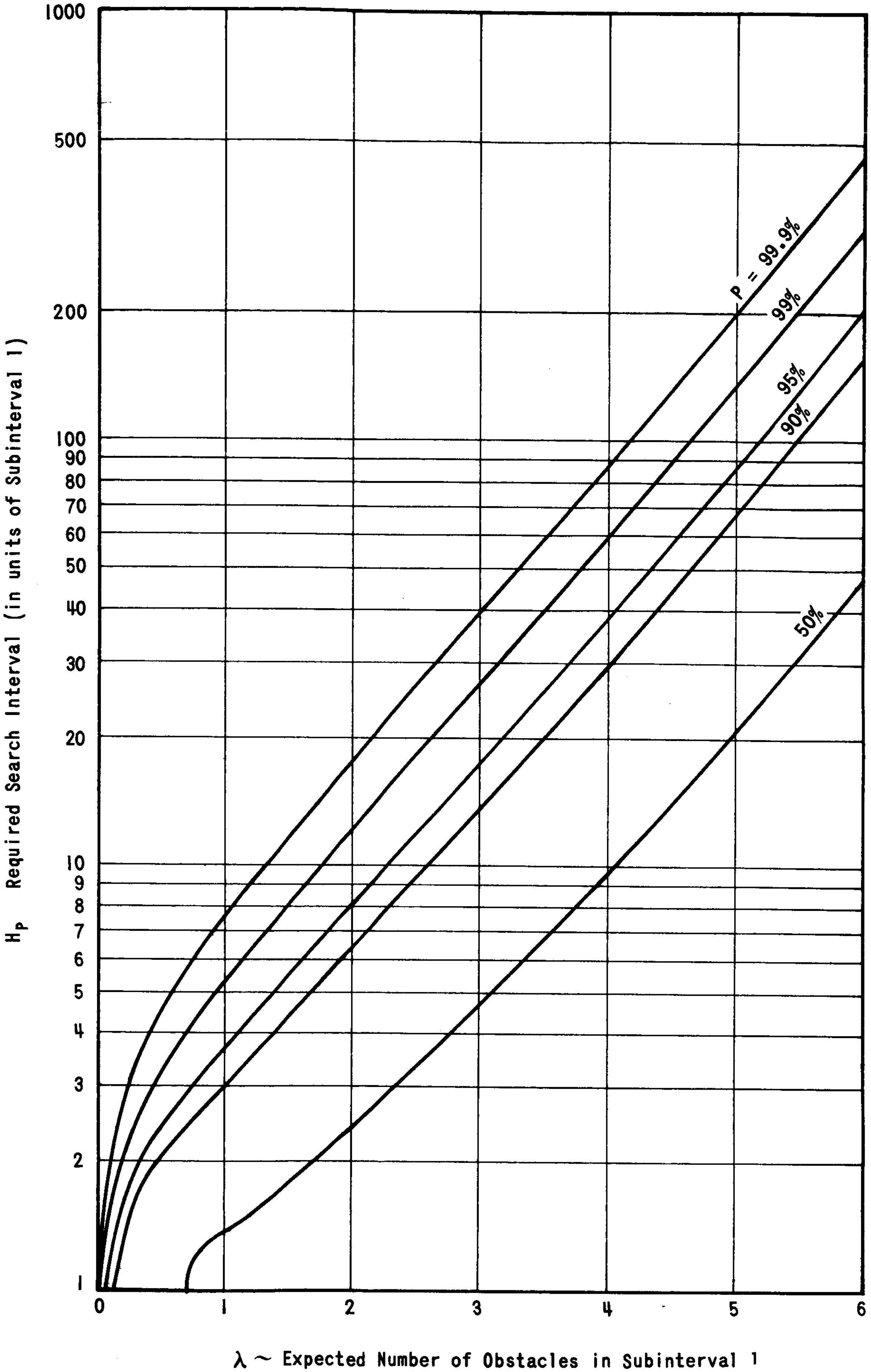

FIGURE 3 - SEARCH INTERVAL REQUIRED FOR P% CONFIDENCE

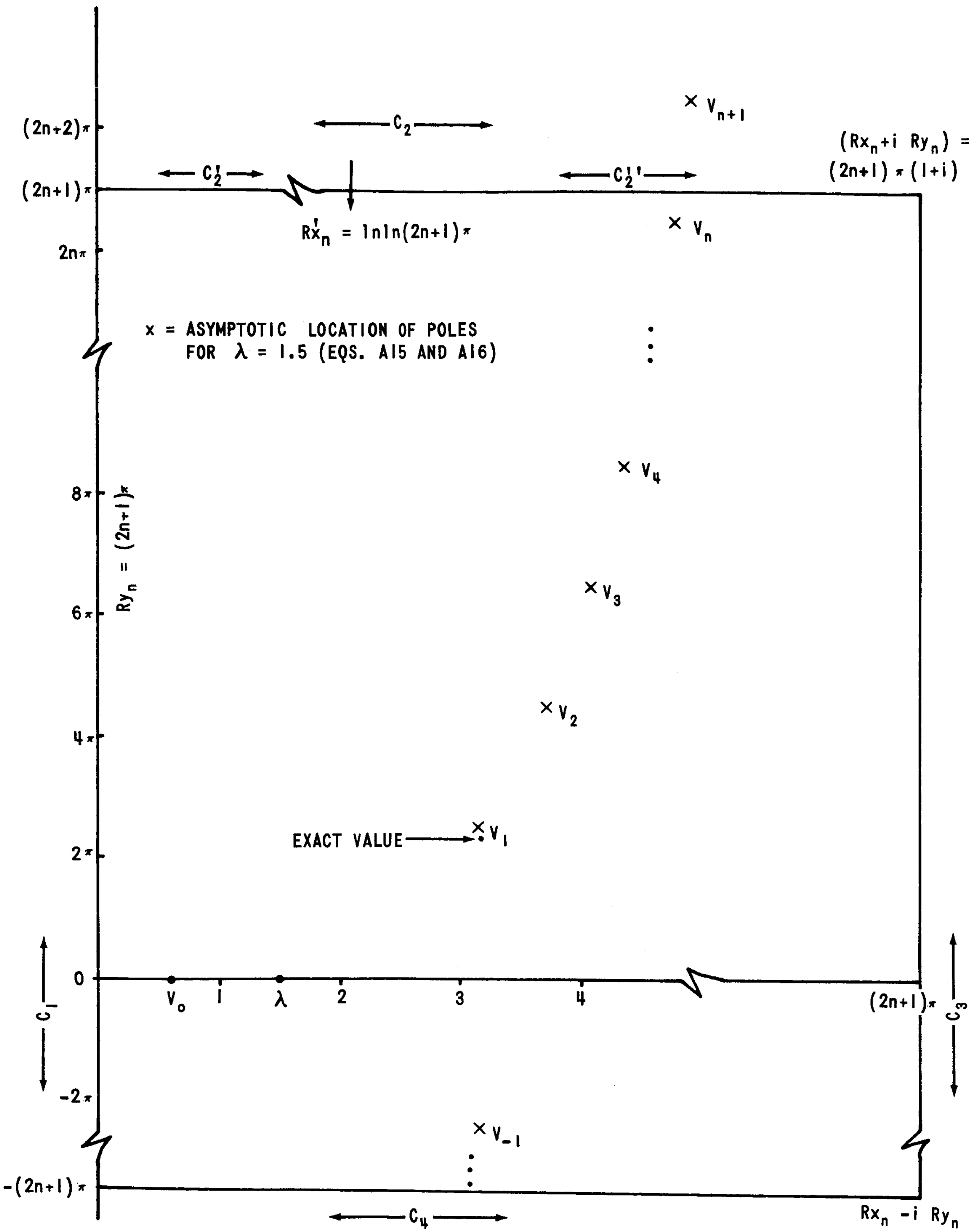

FIGURE B1 - BOUNDARY FOR CONTOUR INTEGRATION